Use the separation of variables procedure on the angular equation

$$\mathbf{L}^2 Y(\theta,\phi) = A\hbar^2 Y(\theta,\phi) \tag{1}$$

$$\text{where } \mathbf{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{2}$$

to obtain the following two equations for the polar and azimuthal angles:

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - B\frac{1}{\sin^2\theta}\right]\Theta(\theta) = -A\Theta(\theta) \tag{3}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -B\Phi(\phi) \tag{4}$$

where A and B are constants.

Solution Ansatz:
Assume the solution if of the form of a product of an unknown function of θ and and unknown function of ϕ:

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

and plug the ansatz into the differential equation. The derivatives wrt θ act only on the function Θ and the derivatives wrt ϕ act only on the function Φ.

$$0 = \overset{1}{\cancel{-\hbar^2}}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + A\right]Y(\theta,\phi) \tag{5}$$

$$= \Phi(\phi)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Theta(\theta) + \Theta(\theta)\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Phi(\phi) + A\Theta(\theta)\Phi(\phi) \tag{6}$$

Divide by $\Theta(\theta)\Phi(\phi)$:

$$0 = \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Theta(\theta) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Phi(\phi) + A \tag{7}$$

Multiply by $\sin^2\theta$ to isolate the different variable in different terms:

$$0 = \frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Theta(\theta) + \frac{1}{\Phi}\frac{\partial^2}{\partial\phi^2}\Phi(\phi) + A\sin^2\theta \tag{8}$$

Now, all of the dependence of the θ variable is in the first and third terms and all of the dependence on the ϕ variable is in the second term, so we can set these terms separately equal to a constant, $\pm B$:

$$\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\Theta(\theta) + A\sin^2\theta = B \tag{9}$$

$$\frac{1}{\Phi}\frac{d^2\Phi(\phi)}{d\phi^2} = -B \tag{10}$$

Finally, some simple rearrangement of the terms yields

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - B\frac{1}{\sin^2\theta}\right]\Theta(\theta) = -A\Theta(\theta) \tag{11}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} = -B\Phi(\phi) \tag{12}$$