A uniform steady current is flowing, parallel to the axis, through an infinitely long cylindrical shell of inner radius a and outer radius b.

1. $\vec{J} = \alpha\, s^3 \hat{z}$

2. $\vec{J} = \alpha\, \frac{\sin ks}{s} \hat{z}$

3. $\vec{J} = \alpha\, e^{ks^2} \hat{z}$

4. $\vec{J} = \alpha\, \frac{e^{ks}}{s} \hat{z}$

For your group's case, answer each of the following questions:

1. Find the total current through a gate which is a cross-section of the wire perpendicular to the current.

Solution

$$I_{total} = \int \vec{J} \cdot \hat{n}\, dA$$
$$\hat{n}\, dA = \hat{z} s' ds' d\phi'$$

$$I_{total,1} = \frac{2\pi}{5}\alpha \left(b^5 - a^5\right)$$
$$I_{total,2} = 2\pi\alpha \left(-\cos kb + \cos ka\right)$$
$$I_{total,3} = \pi\frac{\alpha}{k} \left(e^{kb^2} - e^{ka^2}\right)$$
$$I_{total,4} = 2\pi\frac{\alpha}{k} \left(e^{kb} - e^{ka}\right)$$

2. Use Ampère's Law to find the magnetic field at each of the three radii below:

   a) $s_1 > b$
   b) $a < s_2 < b$
   c) $s_3 < a$

Solution For each region, I need to identify an Amperian loop to perform my calculation. To get the magnetic field out of the integral, I need to pick an Amperian loop where the magnetic field will be constant at every point on the loop.

The magnetic field has to be perpendicular to the current, so the magnetic field can't have a $\hat{z}$ component.

The magnetic field also can't have a $\hat{s}$, and I can show that with a proof by contradiction. Let's say I'm standing at a point near the cylinder with my feet pointed toward the current and I see the current pointing in front of me. I suppose that the magnetic field due to this configuration of current points upward. Then I close my eyes, spin around, and I see the opposite: the current is pointing toward me. I'd therefore would expect that the magnetic field would also flip and now be pointing down. But I haven't moved! And the magnetic field can't be both pointing up and pointing down at the same spot. So, there can't be a $\hat{s}$ component on the magnetic field. At most, the magnetic field can have a $\hat{\phi}$ component.

Similarly, the magnetic field can at most only depend on the coordinate s. The current density isn't different if I translate in the z coordinate or the ϕ coordinate. Therefore, the magnetic field should also not change with those coordinates.

Therefore, $\vec{B} = B_\phi(s)\hat{\phi}$, and I should put my Amperian loop in the $z = 0$ plane centered at the center of the cylinder, with $\hat{n}\,dA = \hat{z}s'ds'd\phi'$ (for calculating the current enclosed by the loop) and $d\vec{\ell} = s\,ds\,d\phi$ for calculating the circulation of the magnetic field.

$$\mu_0 I_{\text{enclosed}} = \oint_{\text{Amperian Loop}} \vec{B} \cdot d\vec{r}$$

Interior to the cyclindrical shell, there is the current enclosed by the loop:

$$\begin{aligned} 0 &= \int_0^{2\pi} B_\phi(s)\hat{\phi} \cdot \hat{\phi} s_1 \, d\phi \\ &= B_\phi(s) \int_0^{2\pi} s_1 \, d\phi \\ \therefore B_\phi(s) & \end{aligned}$$

because the circumference of the Amperian loop is not zero.

3. Sketch the magnitude of the magnetic field as a function of s.