

Find the upward pointing flux of the electric field $\vec{E} = E_0 z \hat{z}$ through the part of the surface $z = -3s^2 + 12$ (cylindrical coordinates) that sits above the (x, y) -plane.

Break this problem into steps:

- Sketch the paraboloid and the vector field.
- Sketch a representative $d\vec{A}$ and calculate $d\vec{A}$ algebraically.
- Calculate the flux.

Solution We want to compute $\int \vec{E} \cdot \hat{n} dA$ over the surface of the paraboloid given, or if you prefer (I find it even simpler to think of this way), just $\int \vec{E} \cdot d\vec{A}$. We know $\vec{E}$, so we really just need to draw little “patches” of area $d\vec{A}$ on the surface, and figure out what those area vectors look like. That’s not so bad—just draw a little patch somewhere on the surface! Mine is a little square which is $d\vec{r}_1$ by $d\vec{r}_2$. Here I really need a picture, but let’s try to visualize it with words:

Let $d\vec{r}_1$ point in the $\hat{\phi}$ direction, with length $s d\phi$ (Do you see why?) That’s the “bottom” leg of a little patch, which sweeps around in the $\hat{\phi}$ direction as you change ϕ just a little. It has no z component, and in fact no radial component either, because this paraboloid sweeps symmetrically around the z -axis.

The other leg has to be “tilted” up (along) the paraboloid surface. I drew myself a little picture and noticed that this vector points “up the paraboloid” (in order that our cross product will give the outward normal), so it would be written

$$d\vec{r}_2 = ds \hat{s} + dz \hat{z} \quad (1)$$

(Draw it, make sure this makes sense to you!) Do **NOT** add an explicit minus sign on the radial component. The limits of integration will take care of this. Now, “use what you know.” $d\vec{r}_2$ lives on a parabola, so s and z are related by $z = -3s^2 + 12$, so $dz = -6s ds$. Decide whether you want to integrate with respect to s or z . I will choose s , so I need to change all the z ’s to s ’s, (including the z in the expression for the electric field!) Therefore, $d\vec{r}_2$ becomes

$$d\vec{r}_2 = (\hat{s} - 6s \hat{z}) ds \quad (2)$$

Now we’re all set.

$$\begin{aligned} d\vec{A} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= s d\phi \hat{\phi} \times (\hat{s} - 6s \hat{z}) ds \\ &= (-s \hat{z} - 6s^2 \hat{s}) ds d\phi \end{aligned} \quad (3)$$

We only CARE about the $\hat{z}$ component of this expression, because we’re about to dot it with $\vec{E} = E_0 z \hat{z}$, and thus we find

$$\vec{E} \cdot d\vec{A} = E_0(-3s^2 + 12)(-s) ds d\phi \quad (4)$$

Almost done! This is our integrand, and we still need to do the double integral. The $d\phi$ integration is sweet, we sweep around the circle and get 2π , since nothing depends on ϕ . So we just have one simple single integration to do, along ds , which should run from 2 to 0. (Oh, do you see why that is 2? This surface is like a hat, and reaches $z = 0$ at the point $s = 2$.) Make sure to start at $s = 2$, the outside of the hat, and run to $s = 0$ at the top of the hat because that is the direction that $d\vec{r}_2$ points.

Therefore, the integral that we have to do is:

$$\begin{aligned}\int \vec{E} \cdot d\vec{A} &= \int_2^0 \int_0^{2\pi} E_0(3s^3 - 12s) ds d\phi \\ &= 24\pi E_0\end{aligned}\tag{5}$$

By the way, you can check this answer using the divergence theorem, which we'll talk about later in the week! The surface integral of $\vec{E} \cdot d\vec{A}$ must equal the volume integral of the divergence of E , (where $\vec{\nabla} \cdot \vec{E} = E_0$ in this case). So we just need the volume of the paraboloid. I leave that as an exercise for you (just slice it into circular “plates”); it works! One thing to watch out for is that the surface integral needs to include the flux through the BOTTOM of the paraboloid, too. (The divergence theorem only works for **closed** surfaces.) But we're ok, because $z = 0$ there, and so there is no extra flux from that surface.