1 Simultaneous equations with differentials

When working with differentials, the trick is generally to use linear algebra to solve for the differential you want in terms of the differentials you want to see it related to. You can always do this because differentials equations are linear in the differentials, i.e. the differentials only occur to the first power, and are not inside functions.

1.1 A symbolic example

Let's consider a symbolic example. Suppose you are given the following two equations of state describing the behavior of a PDM.

$$F_L^2 + F_R^2 = k \sinh\left(\frac{x_L}{x_R}\right) \tag{1}$$

$$x_L - x_R = \cos\left(1 + \frac{F_L}{F_R}\right) \tag{2}$$

Note that since we have two independent variables, we know that there must exist two equations that determine the two remaining dependent variables.

Let us ask ourselves a simple question. How stiff is the first string when the other string is held fixed? Mathematically this stiffness would be something like

$$\left(\frac{\partial F_L}{\partial x_L}\right)_{x_R} \tag{3}$$

which if you examine it looks like a stiffness, in the sense that it measures how much force is required to move the string by a little bit. You could also think of this as a spring constant for small perturbations of the string.

How would we find this? We would want an expression for dF_L in terms of the differentials dx_L and dx_R.

We can start by zapping each of our equations with d to obtain two equations of differentials. (You will often start problems in this way, since zapping with d is almost always the easiest thing to do.)

$$2F_L\,dF_L + 2F_R\,dF_R = k\cosh\left(\frac{x_L}{x_R}\right)\left(\frac{dx_L}{x_R} - \frac{x_L}{x_R^2}dx_R\right) \tag{4}$$

$$dx_L - dx_R = -\sin\left(1 + \frac{F_L}{F_R}\right)\left(\frac{dF_L}{F_R} - \frac{F_L}{F_R^2}dF_R\right) \tag{5}$$

These look a bit messy, but in planning our attack, all we need to keep in mind is that we are looking for an equation that involves *only* the differentials dx_L, dx_R, and dF_L. In other words, all we need to do is to eliminate dF_R from one of these equations. We can do that by solving for dF_R and substituting.

$$dF_R = \frac{k\cosh\left(\frac{x_L}{x_R}\right)\left(\frac{dx_L}{x_R} - \frac{x_L}{x_R^2}dx_R\right) - 2F_L\,dF_L}{2F_R} \tag{6}$$

$$= \frac{k}{2F_R x_R}\cosh\left(\frac{x_L}{x_R}\right)\left(dx_1 - \frac{x_L}{x_R}dx_R\right) - \frac{F_L}{F_R}dF_L \tag{7}$$

$$dx_L - dx_R = -\sin\left(1+\frac{F_L}{F_R}\right)\left(\frac{dF_L}{F_R} - \frac{F_L}{F_R^2}\left(\frac{k}{2F_Rx_R}\cosh\left(\frac{x_L}{x_R}\right)\left(dx_L - \frac{x_L}{x_R}dx_R\right) - \frac{F_L}{F_R}dF_L\right)\right) \tag{8}$$

Note that although we have simply to plugged Eq. 7 into Eq. 5 to obtain Eq. 8. It is big and messy-looking, but doesn't involve anything complicated. It is often helpful to gather together each differential, so each differential (in this case dx_L, dx_R and dF_L) appears just once in the equation.

$$\left(1 - \frac{kF_L}{2F_R^3x_R}\cosh\left(\frac{x_L}{x_R}\right)\sin\left(1+\frac{F_L}{F_R}\right)\right)dx_L - \left(1 - \frac{kF_Lx_L}{2F_R^3x_R^2}\cosh\left(\frac{x_L}{x_R}\right)\sin\left(1+\frac{F_L}{F_R}\right)\right)dx_R = \\ -\sin\left(1+\frac{F_L}{F_R}\right)\left(\frac{1}{F_R}+\frac{F_L^2}{F_R^3}\right)dF_L \tag{9}$$

To find the derivative we seek, we just need to solve for dF_L.

$$dF_L = -\frac{1 - \frac{kF_L}{2F_R^3x_R}\cosh\left(\frac{x_L}{x_R}\right)\sin\left(1+\frac{F_L}{F_R}\right)}{\sin\left(1+\frac{F_L}{F_R}\right)\left(\frac{1}{F_R}+\frac{F_L^2}{F_R^3}\right)}dx_L + \frac{1 - \frac{kF_Lx_L}{2F_R^3x_R^2}\cosh\left(\frac{x_L}{x_R}\right)\sin\left(1+\frac{F_L}{F_R}\right)}{\sin\left(1+\frac{F_L}{F_R}\right)\left(\frac{1}{F_R}+\frac{F_L^2}{F_R^3}\right)}dx_R \tag{10}$$

Now that we have the differential dF_1 expressed in terms of just the two differentials dx_1 and dx_2 (since we have two independent variables in this system), we can simply read of the stiffness derivative we are seeking as the coefficient in front of the dx_1 differential:

$$\left(\frac{\partial F_L}{\partial x_L}\right)_{x_R} = -\frac{1 - \frac{kF_L}{2F_R^3x_R}\cosh\left(\frac{x_L}{x_R}\right)\sin\left(1+\frac{F_L}{F_R}\right)}{\sin\left(1+\frac{F_L}{F_R}\right)\left(\frac{1}{F_R}+\frac{F_L^2}{F_R^3}\right)} \tag{11}$$

This is a good time to remind yourself why this particular coefficient is this particular partial derivative. When you read Eq. 10, consider what happens if you set to zero one of the differentials on the right-hand side, in this case dx_R. This represents holding fixed x_R. Then you ask (with this held fixed) what is the ratio between the small change in F_L and the small change in x_L. This ratio is the coefficient we found, which must therefore be the partial derivative with x_R held fixed.

It is possible that we have not convinced you by this example that this is the easy way to find this partial derivative. There are other ways to solve for this stiffness, and of course innumerable ways to simplify this expression. we have chosen this rather tedious and artificial example to highlight that once you have zapped with d and obtained an equation relating differentials, the rest is simply linear algebra to express the differential you seek in terms of the differentials you wish to relate it to. You will practice this many times, both in class and in your homework. In many cases, rather than working with explicit analytic expressions, you will derive relationships that are true for arbitrary equations of state (e.g. chain rules).