An astronomer is studying an interesting region of interstellar medium. The interstellar medium is a gas consisting primarily of atomic hydrogen. The energy eigenstates of a hydrogen atom are enumerated by three integers:

$$n \geq 1 \tag{1}$$
$$0 \leq \ell < n \tag{2}$$
$$-\ell \leq m \leq \ell \tag{3}$$

The energy eigenvalues are

$$E_{n\ell m} = -\frac{13.6 \text{ eV}}{n^2} \tag{4}$$

The astronomer measures the fraction of atoms in different energy eigenstates and finds that

$$\frac{N_{1s}}{N_{2p}} = 10^5 \tag{5}$$

where N_{1s} is the number of atoms in the $1s$ state (i.e. $n = 1, \ell = 0, m = 0$) and N_{2p} is the number of atoms in states with $n = 2$ and $\ell = 1$. Solve for the temperature in this region of the interstellar medium.

Solution We start by recognizing that the number of atoms in a given state is proportional to the probability of any given atom being in that state. However our N_{2p} is not the number of atoms in a given state, but rather in any one of three states, which have $m = -1, 0, 1$. Thus the ratio

$$\frac{N_{1s}}{N_{2p}} = \frac{P_{1s}}{P_{2p}} \tag{6}$$
$$= \frac{\frac{e^{-\beta E_{1s}}}{Z}}{3\frac{e^{-\beta E_{2p}}}{Z}} \tag{7}$$

where in the last step, I used the Boltzmann probabilities with a factor of three on the bottom to account for the three different states, each of which has a probability of $\frac{e^{-\beta E_{2p}}}{Z}$. So now we just have some algebra to do. And then we'll have some unit conversions to do.

$$\frac{N_{1s}}{N_{2p}} = \frac{\frac{e^{-\beta E_{1s}}}{\cancel{Z}}}{3\frac{e^{-\beta E_{2p}}}{\cancel{Z}}} \tag{8}$$
$$= \frac{1}{3}e^{-\beta(E_{1s}-E_{2p})} \tag{9}$$
$$= \frac{1}{3}e^{-\beta\left(-R_y-\frac{R_y}{4}\right)} \tag{10}$$
$$= \frac{1}{3}e^{\beta\frac{3}{4}R_y} \tag{11}$$

$$\ln\left(\frac{N_{1s}}{N_{2p}}\right) = \beta\frac{3}{4}R_y - \ln 3 \tag{12}$$

$$\begin{aligned} T &= \frac{3R_y}{4k_B \ln\left(3\frac{N_{1s}}{N_{2p}}\right)} && (13) \\ &= \frac{3 \cdot 13.6 \text{ eV}}{4 \cdot \frac{1}{12}\frac{\text{meV}}{\text{K}} \cdot \frac{1 \text{ eV}}{10^3 \text{ eV}} \ln(3 \times 10^5)} && (14) \\ &\approx 10,000 \text{ K} && (15) \end{aligned}$$

So, this bit of the interstellar medium is pretty hot. Some parts are a lot hotter than this, and some are much colder. Wikipedia has a nice article about H II regions that have this temperature: H II regions.