For the state

$$|\Psi\rangle = \sqrt{\frac{7}{10}}|2,1,0\rangle + \sqrt{\frac{1}{10}}|3,2,1\rangle + i\sqrt{\frac{2}{10}}|3,1,1\rangle$$

Calculate

- $\mathcal{P}(L_z = \hbar)$

Solution

$$\mathcal{P}_{L_z=\hbar} = |\langle 3,2,1|\Psi\rangle|^2 + |\langle 3,1,1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \tag{1}$$

$$\tag{2}$$

- $\langle L_z \rangle$

Solution There are a number of viable ways to find an expectation value, if the probabilities are easy to read off, I think summing the probabilities times the eigenvalues you care about is the easest. We can read off the probabilities:

$$\mathcal{P}_{L_z=\hbar} = |\langle 3,2,1|\Psi\rangle|^2 + |\langle 3,1,1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \tag{3}$$

$$\mathcal{P}_{L_z=0\hbar} = |\langle 2,1,0|\Psi\rangle|^2 = \left|\sqrt{\frac{7}{10}}\right|^2 = \frac{7}{10} \tag{4}$$

Now we can just take the weighted average:

$$\langle L_z \rangle = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} P_{L_z=m\hbar} m\hbar = \left(\frac{7}{10}\right)(0\hbar) + \left(\frac{3}{10}\right)(\hbar) = \frac{3}{10}\hbar \tag{5}$$

Then, if you have time, continue with these calculations:

- $\mathcal{P}(L^2 = 2\hbar^2)$

Solution

$$\mathcal{P}_{L^2=2\hbar^2} = |\langle 2,1,0|\Psi\rangle|^2 + |\langle 3,1,1|\Psi\rangle|^2 = \left|\sqrt{\frac{7}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{7}{10} + \frac{2}{10} = \frac{9}{10} \tag{6}$$

$$\tag{7}$$

- $\langle L^2 \rangle$

Solution Here we use the same strategy, but need to be aware of different degeneracies:

$$\mathcal{P}_{L^2=2\hbar^2} = |\langle 2,1,0|\Psi\rangle|^2 + |\langle 3,1,1|\Psi\rangle|^2 = \left|\sqrt{\frac{7}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{7}{10} + \frac{2}{10} = \frac{9}{10} \tag{8}$$

$$\mathcal{P}_{L^2=6\hbar^2} = |\langle 3,2,1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 = \frac{1}{10} \tag{9}$$

Now we can just take the weighted average:

$$\langle L^2 \rangle = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} P_{L^2=\ell(\ell+1)\hbar^2}\ell(\ell+1)\hbar^2 = \left(\frac{9}{10}\right)\left(2\hbar^2\right) + \left(\frac{1}{10}\right)\left(6\hbar^2\right) = \frac{24}{10}\hbar^2 \tag{10}$$

- $\mathcal{P}(E = -13.6eV/3^2 = -1.51eV)$

Solution

$$\mathcal{P}_{E=-13.6eV/9} = |\langle 3,2,1|\Psi\rangle|^2 + |\langle 3,1,1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \tag{11}$$

$$\tag{12}$$

- $\langle E \rangle$

Solution

$$\mathcal{P}_{E=-13.6eV/9} = |\langle 3,2,1|\Psi\rangle|^2 + |\langle 3,1,1|\Psi\rangle|^2 = \left|\sqrt{\frac{1}{10}}\right|^2 + \left|i\sqrt{\frac{2}{10}}\right|^2 = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \tag{13}$$

$$\mathcal{P}_{E=-13.6eV/4} = |\langle 2,1,0|\Psi\rangle|^2 = \left|\sqrt{\frac{7}{10}}\right|^2 = \frac{7}{10} \tag{14}$$

Now we can just take the weighted average:

$$\langle E \rangle = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} P_{E=-13.6eV/n^2}\left(-\frac{13.6eV}{n^2}\right) = \left(\frac{7}{10}\right)\left(-\frac{13.6eV}{4}\right) + \left(\frac{3}{10}\right)\left(-\frac{13.6eV}{9}\right) = -2.83eV \tag{15}$$

- What measurements can be degenerate on the Hydrogen atom?

Solution Measurements of the z-component of angluar momentum, total angular momentum, and energy can all be degenerate on the hydrogen atom. When we have 3 quantum numbers we can always lock one down and freely change the other two to create different eigenstates which will be degenerate for the locked quantum number.

- What is the time development of this state?

Solution Since the hydrogen atom energy eigenvalues depend only on n,

$$E_n = -\frac{13.6eV}{n^2}, \tag{16}$$

each energy eigenstate picks up a time-dependent phase factor $e^{-iE_n t/\hbar}$. Therefore,

$$|\Psi(t)\rangle = \sqrt{\frac{7}{10}}e^{-iE_2 t/\hbar}|2,1,0\rangle + \sqrt{\frac{1}{10}}e^{-iE_3 t/\hbar}|3,2,1\rangle + i\sqrt{\frac{2}{10}}e^{-iE_3 t/\hbar}|3,1,1\rangle. \tag{17}$$

Using

$$E_2 = -\frac{13.6eV}{4}, \qquad E_3 = -\frac{13.6eV}{9}, \tag{18}$$

we can also write

$$|\Psi(t)\rangle = \sqrt{\frac{7}{10}}e^{-iE_2 t/\hbar}|2,1,0\rangle + e^{-iE_3 t/\hbar}\left(\sqrt{\frac{1}{10}}|3,2,1\rangle + i\sqrt{\frac{2}{10}}|3,1,1\rangle\right). \tag{19}$$

Notice that the two $n = 3$ states have the same energy, so they acquire the same overall time-dependent phase.

- What is the probability of finding the particle in the region $0 < \theta < \pi/6$, $\pi/3 < \phi < \pi/2$, and $r_1 < r < r_2$?

Solution To find the probability of locating the particle in a particular region of space, we need to integrate the probability density over that region:

$$\mathcal{P} = \int_{r_1}^{r_2}\int_0^{\pi/6}\int_{\pi/3}^{\pi/2} |\Psi(r,\theta,\phi,t)|^2 r^2 \sin\theta \, d\phi \, d\theta \, dr. \tag{20}$$

For this state, the position-space wavefunction is

$$\Psi(r,\theta,\phi,t) = \sqrt{\frac{7}{10}}e^{-iE_2 t/\hbar}R_{21}(r)Y_1^0(\theta,\phi) \tag{21}$$

$$+ \sqrt{\frac{1}{10}}e^{-iE_3 t/\hbar}R_{32}(r)Y_2^1(\theta,\phi) \tag{22}$$

$$+ i\sqrt{\frac{2}{10}}e^{-iE_3 t/\hbar}R_{31}(r)Y_1^1(\theta,\phi). \tag{23}$$